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INTUITION IN SCIENCE AND MATHEMATICS

An Educational Approach

INTUITION IN SCIENCE AND MATHEMATICS

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PART II: FACTORS WHICH SHAPE INTUITIONS

PREFACE

In writing the present book I have had in mind the following objectives:

- To propose a theoretical, comprehensive view of the domain of intuition.
- To identify and organize the experimental findings related to intuition scattered in a wide variety of research contexts.
- To reveal the educational implications of the idea, developed for science and mathematics education.

Most of the existing monographs in the field of intuition are mainly concerned with theoretical debates - definitions, philosophical attitudes, historical considerations. (See, especially the works of Wild (1938), of Bunge (1962) and of Noddings and Shore (1984).)

A notable exception is the book by Westcott (1968), which combines theoretical analyses with the author's own experimental studies.

But, so far, no attempt has been made to identify systematically those findings, spread throughout the research literature, which could contribute to the deciphering of the mechanisms of intuition. Very often the relevant studies do not refer explicitly to intuition. Even when this term is used it occurs, usually, as a self-evident, common sense term.

The explanation is that intuition is generally seen as a primary phenomenon which may be described but which is not reducible to more elementary components. As a matter of fact, intuitively, intuition has the appearance of a self-evident, self-consistent cognition, like the perception of a color or the experience of an emotion. The effect is that, generally, no attempt is made by researchers to use their experimental findings for elucidating the structure of intuitive phenomena. On the contrary: it is intuition which is used as a descriptive and explanatory concept. Piaget, for instance, who used the terms intuition and intuitive thinking very often never made any attempt to refer to his own findings for obtaining a deeper understanding of the general structure of intuitive mechanisms as such. With a very few exceptions this is the situation with most of the researchers. Such an exception is the work of Andrea DiSessa (1982, 1983a, 1983b) who made an important attempt to analyse his own experimental findings with the explicit purpose of building a theory of intuition.

The present book has a similar aim. We expound a theory, we try to get a richer insight in the mechanisms of intuition using research evidence, and we try to support and enlarge our conception on the basis of these data.

Our general thesis is that, in principle, cognition fulfils behavioral aims and is shaped by behavioral constraints. The same is to be said about intuition which is a particular form of cognition.

Intuition has its roots in the syncretic type of thinking of the child and of human beings in the early stages of civilization. But it does not survive in adults and in highly developed cultures only as a mere residuum. We claim that intuition expresses a profound necessity of our mental behavior.

During the very course of our reasoning, of our trial-and-error attempts, we have to rely on representations and ideas which appear, subjectively, as certain, self-consistent and intrinsically clear. We cannot doubt everything at every moment. This would be a paralysing attitude. Some representations, some conceptions have to be taken for granted. They have to appear, subjectively, as autonomous, coherent, totally and directly acceptable cognitions in order to keep the process of reasoning working fruitfully.

An intuition is, then, such a crystallized - very often prematurely closed - conception in which incompleteness or vagueness of information is masked by special mechanisms for producing the feelings of immediacy, coherence and confidence.

Such mechanisms have been described in the research literature, but very often without any apparent connection with a theory of intuition. In the present work an attempt has been made to take advantage of these research sources. Studies in overconfidence, in subjective probabilities, findings referring to mental models, to typical errors in naive physics, to misconceptions in mathematics, to the evolution of logical concepts in children etc. represent, in fact, rich potential sources for a theory of intuition.

I should like to emphasize again this interesting phenomenon: simply because intuition is tacitly but firmly considered to be a primitive feeling, rich sources of information, based on experimental findings, have been ignored by most of the theorists. A primary purpose of this book has been to overcome this obstacle.

The history of science and mathematics is also an important source for understanding the dramatic struggle of the scientific mind against intuitive biases. We have used a number of examples in our work in this respect, trying to identify the common structure of intuitive difficulties in experts and novices.

Turning our attention, now, to the educational aspects: Most authors - experimental researchers and theorists alike - strive to set up recommendations for avoiding intuitively-based errors in learning and problem solving and for improving intuitive guesses and evaluations. But, in our opinion, one can hardly expect such suggestions to be really helpful if they are not based on a comprehensive theory of intuition. Intuitions are only apparently autonomous, self-evident cognitions. They are so in order to confer on some of the individual's ideas the appearance of certitude and intrinsic validity. But, in fact, these ideas appear very robust as an effect of their being deeply rooted in the person's basic mental organization. Consequently, in order to eliminate or change or even control an intuitive attitude it would be necessary to produce a profound, structural transformation in large areas of mental activity

Therefore, the individual's self-confidence itself might be endangered if he learns that even his deepest beliefs may very often be misleadiag. It follows that intuitions cannot be treated effectively and positively as mere isolated symptoms but rather as manifestations of highly articulated and very complex structures. And for this, a comprehensive theory is necessary, a theory which would take into acount the behavioral roots and the adaptive functions of intuitions.

I do not pretend to have solved all these puzzling theoretical problems. It is probably too early to get definite answers. But, if I have been able to convince the reader of the importance of this line of thought, if I have been able to cast some doubts and to produce some new expectations with regard to his intuition about intuition then my main goal will have been achieved.

The book is divided into two major parts. The first part is concerned with the theoretical aspects - the theory, the relevance of intuitive forms of cognition for scientific and mathematical reasoning, the connections between intuition and other, related, categories of cognition, the general characteristics and the classification of intuitions.

The second part of the book deals mainly with factors which contribute to shaping intuitions: the role of experience, the role of various types of models - analogies, paradigms, diagrams, phenomenological primitives - the role of factors for producing the effects of immediacy and globality.

I believe that the time is now ripe for a formal recognition of intuition as one of the major components of our cognitive endeavors. A constant interplay between theory and experiment in this field is nowadays certainly possible and highly desirable.

It is our conviction that the findings will be beneficial for both cognitive psychology and educational practice.

ACKNOWLEDGMENTS

For the past ten years I have given lectures and directed seminars on intuition at the School of Education of Tel Aviv University. This book is largely based on that activity. My gratitude goes, then, first to my students and those colleagues who patiently helped me during all these years to clarify my ideas and to see new problems where everything seemed to be settled.

I am very grateful to Alan Bishop for inviting me to write this book and for encouraging me to continue my work whenever he felt that the rhythm was slowing down.

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PART I

THE THEORY

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CHAPTER 1

INTUITION AND THE NEED FOR CERTITUDE

THE COMPLEXITY OF THE DOMAIN

Intuition is certainly a highly controversial concept in science and philosophy. Accepted by some as the basic source of every true knowledge, rejected by others as potentially misleading every quest for truth, intuition - as a concept and as a method - revives itself again and again in philosophical disputes, in the theoretical foundations of science and mathematics, in mystical considerations, in ethics and aesthetics, in pedagogy, and yet very little and very seldom in psychology.

The Variery of Meanings

In some contexts, intuition is referred to as a source of true - or apparently true - knowledge. It is generally in this sense that the term intuition is used in the works of Descartes (1967) and Spinoza (1967). For both of them, in a world of misleading appearances and futile interpretations, intuition remains the ultimate reliable source of absolutely certain truths.

For others, intuition is rather a method, a sort of mental strategy which is able to reach the essence of phenomena. Bergson has been the main advocate of this usage. According to Bergson, intelligence addresses itself to the world of objects, of solids, of static realities. In order to understand reality, intelligence uses a "cinematographic" procedure: the uninterrupted flow of real phenomena is cut into sequences of static representations mainly expressed in concepts. But the essence of motion, of life, of spirit, of duration cannot be reached this way. According to Bergson it is through intuition - a kind of sympathetic identification - that we are able to grasp the very essence of living and changing phenomena (Bergson, 1954).

The term intuition is also used for indicating a certain category of cognitions, i.e. cognitions which are directly grasped without, or prior to, any need for explicit justification or interpretation. It is in this sense that Piaget refers to spatial and temporal intuitions, to empirical and operational intuitions, to pure intuitions etc. (Beth and Piaget, 1961).

In Kant's terminology the concept of intuition gets a more restrictive meaning, compared with those referred to in the previous presentations. Namely, according to Kant, intuition is simply the faculty through which objects are directly grasped in distinction to the faculty of understanding through which we achieve conceptual knowledge. Kant uses the terms intellectual and sensible intuitions, but practically, it is only the sensible variant which makes sense to him. An "intellectual intuition" would be necessary for knowing the "noumenon", the reality in itself - and this is impossible. Therefore in Kant's terminology intuition remains related to sensorial knowledge, while an ''intellectual" intuition simply does not exist (Kant, 1980, p. 268).

The term "intellectual intuition" may also be used for designating forms of immediate knowledge which are not sensorial, which deal with concepts, formal relations, theories. One may affirm, for instance, that the statement: "Every natural number has a successor" is intuitively acceptable, and in this case we have an "intellectual" intuition. In contrast, the intuitive evaluation of the weight of an object or of the speed of a moving body would represent sensorial intuitions. Certainly, no clear-cut distinction is possible, but the terminology related to intuition is so confusing that we feel that even in these introductory lines the reader should get a preliminary picture of the complexity of this domain.

As has already been noticed with regard to the credibility of intuitions, we may also identify various, even opposite, conceptions. Sometimes, intuition is referred to as a global guess for which an individual is not able to offer a clear and complete justification. Very often intuition means an elementary, common sense, popular, primitive form of knowledge, as opposed to scientific conceptions and interpretations. In contrast, according to some philosophers, like Spinoza, intuition is the highest form of knowledge through which the very essence of things, and God Himself, may be revealed. According to Poincaré, no genuine creative activity is possible in science and in mathematics without intuition, while for Hahn (1956) intuition is mainly a source of misconceptions and should be eliminated from a serious scientific endeavor.

In the pedagogical literature, intuition is often related to sensorial knowledge as the first necessary basis for a further intellectual education. In this sense, intuitive knowledge is, more or less, equivalent to perceptual knowledge (i.e. concrete objects, pictures, diagrams). In some educational approaches one advocates the necessity to use a large amount of intuitive (concrete, pictorial, manipulative) devices, while according to others one has to eliminate, as soon and as far as possible, intuitive techniques, especially when considering an abstract domain like mathematics.

The term intuition also has special connotations in particular domains. One speaks of "moral intuition" which would represent an a priori knowledge of the notions of "right" and "wrong" (Wild, 1938, p. 131).

In the philosophy of Benedetto Croce intuition plays an essential role in aesthetic feelings. According to Croce, beauty is not a property of Nature. It is rather the product of a specific kind of selection and synthesis which is accomplished by the human mind through intuition. As a matter of fact, in Croce's view intuition is always associated with the sense of beauty, because intuition is always associated with unity in a multiplicity of appearances (See Wild, 1938, pp. 39—49).

Since to some philosophers intuition is the way to reach the essence; the absolute truth, a natural consequence would be to consider intuition as the way to approach divinity. Mystical and generally religious intuitions have often been discussed m philosophical and theological works (see Wild, 1938, pp. 97–114). Let me also mention the use of the term intuition as related to professional capacities. A physician, an engineer, a politician, a psychologist etc. may be said to be able to use his or her intuition in solving complex professional problems: the solution seems to appear promptly only on the basis of an apparently summary evaluation .

Let me mention the example of Berne, a psychiatrist who has elaborated a theoretical approach to intuition on the basis of his professional experience. According to Berne a specialist becomes able, as a result of practice, to make correct, global, professional evaluations by resorting to a great variety of cues about which he is, in fact, not aware (cf. Westcott, 1968, pp. 42—44).

Related Terms

What complicates the domain of intuition still further is that many other terms are used in reference to the same category of phenomena.

Sometimes people use the term *insight* for indicating a sudden, global rearrangement of data in the cognitive field which would allow a new View, a new interpretation or solution in the given conditions. The terms *revelation* (especially in religious contexts) and *inspiration* (in artistic matters) are also used, sometimes, as synonymous with intuition (at least with some of its meanings).

Very often, ”common sense”, “naive reasoning”, “empirical interpretation” are used in reference to forms of knowledge which may also be considered as equivalent to intuitive knowledge.

M. Reuchlin, a well known French psychologist, has published a very interesting paper devoted to what he has called “la pensée naturelle” (Reuchlin, 1973). “Natural thinking” possesses qualities which distinguish it from formal reasoning, but which play an essential adaptive function: immediacy, concreteness, capacity for sudden and global evaluations.

Piaget uses the French term “self-evidence” in a sense which is very similar to intuitive acceptance. For instance, he writes: “the Michelson and Morley experiment demolished the self-evidence of absolute and universal time” (Beth and Piaget, 1966, p. 194).

Related Areas of Investigation

The domain of intuition and the different and contradictory meaning to which it refers are related to a great variety of cognitive investigations. Let us remember some of them: *Problem solving* (illumination, heuristics, anticipatory schemas etc.); *Images and models* (intuitive representations, intuitive models, intuitive didactical means, thinking in images etc.); *belief* and *levels* *of confidence; developmental stages of intelligence* (Piaget has described intuitive thinking as a preoperational stage).

The Contradictory Domain of Intuition

The attempt to find a common definition for this great variety of meanings, features and connotations seems to represent an impossible task. Intuitive knowledge seems to cover the whole domain of cognition. In Ewing's view, even the formal syllogistic strategies have not, ultimately, any other basis than the intuitive belief in the legitimacy of the restrictive structures.

Wild has described intuition as: ". . . a plant of confused and intricate growth which has wound its tendrils round many noble trees and mingled its roots with those of the brightest flowers and most ineradicable weeds in the philosopher's garden". (Wild, 1938, Preface). Why then not give up and simply eliminate the term intuition from a scientific vocabulary? It seems that, for each of its virtual meanings, there exist other, more specific, terms like common sense, understanding, belief, guess, insight, and indeed, no psychological textbook, as far as I know, has included intuition among the basic concepts with which it deals. Despite this, the term intuition and its derivatives appear very often, even in psychological descriptions, but without conferring on it a formal, scientific status. Intuition is used rather as a common sense term or as a primitive notion.

In fact we are posing, in this descriptive analysis, three different but related questions:

Have these apparently very different phenomena termed as intuitions, some basic, common, features or some "family resemblance" which would justify the acceptance of intuition as a definable concept?

If the answer to the above question is affirmative, how is it possible that, despite these common features, the term intuition reveals such contradictory connotations (as, for instance, the highest, the perfect form of knowledge on the one hand; and an unreliable, potentially misleading, form of knowledge, on the other)?

How is it possible that such a confused, hazy term reappears persistently again and again with a preeminent role in many important domains like philosophy, science, mathematics, ethics, art, religion?

Let me suggest an answer to the first question. There is a basic common feature which, despite striking differences, allows the various meanings to be related in a common conceptual structure. *Intuitive knowledge is immediate knowledge; that is, a form of cognition which seem to present itself to a person as being self-evident.* Therefore, intuitive knowledge may appear, in some texts, as being similar to *sensorial* (perceptual) knowledge. But, at the same time, intuition, as an immediate cognition, may be the source of religious revelations, of artistic inspirations, of scientific illumination etc. In all these instances, one deals with apparently *immediate* forms of cognition.

Why, then, besides this common feature - immediacy - have so many different, contradictory properties been attached to the term intuition (second question)? Why, despite this, has the term intuition survived so long in such a variety of domains and with such a variety of contradictory connotations (the third question)? Would it not be fair to admit that the unique common property - immediacy - is, by itself, too poor for characterizing unequivocally a scientific concept?

THE NEED FOR CERTITUDE

Intuition and Belief

My explanation of the persistent use of the term intuition in many fields and despite the apparent contradictions to which it seems to lead is that intuition expresses, beyond its phenomenological, psychological changing appearance, the natural, almost instinctual belief of every human being in the existence of some ultimate, absolutely reliable, certitudes. In a world of potentially misleading uncertainties our practical decisions cannot rely only upon indirect inferences, on theoretically based suppositions. We feel the fundamental need "to see" with our mind, as we see with our eyes. In order to survive, we have to act in accordance with a given, credible reality. Therefore, credible is synonymous with *behaviorally meaningful.* Non-speaking animals are not bothered by credibility. But language and reasoning have produced a breach in this naturally united structure: cognition and behavior. By way of reasoning we know infinitely more than we know through direct perceptual representations. But, as an effect of indirect (conceptual) forms of knowledge, uncertainty becomes a habitual presence in our decision making processes. In order to overcome it a new form of certitude has been invented, corresponding to the symbolic, indirect forms of knowing. This is formal, logically based, certitude. However, the conceptual, logical structure represents a closed system, *a system which controls only its own internal (mental) products.* Its certitudes may have or may not have some practical relevance. What logic offers by itself is not an absolute, practically valuable certitude but *a conventional form of acceptance.*

It is *the need* for a behavioral, practical, non-conventional, implicitly meaningful certitude which creates the almost instinctual belief in the existence of such ultimate certitudes and, consequently, the *quest* for them. It was probably Descartes who best expressed this view: If knowledge is always the product of an active mind, one has to find in the mind itself the criteria through which a certain truth may be distinguished from uncertain appearances.

It is the basic need for unshakeable, self-sufficient certitudes which in our opinion is expressed in the perpetual (very often unconscious) tendency towards direct (directly credible) evidence. It is the *same need* which manifests itself in religious revelations, in the scientists' quest for. intuitive . models or in the mathematician's endeavor to "see".- at *the end of a tremendous analytical effort* - the solution to a problem as a unique global directly acceptable intrinsically meaningful structure.

Apparently, instrospectively - . and this View has been consistently expressed by Descartes and Spinoza - one may get the idea that intuitive revelations (experienced as an intrinsic belief) represent the absolute source and guarantee of certitude. .

It is the absolute need and the almost instinctual quest for certitude which, historically and psychologically, have shaped this particular type of information processing. Disparate or incomplete data agglutinate themselves through it in apparently coherent, consistent compact, intrinsically ctedible structures. They appear subjectively as directly reliable landmarks, indispensable for the continuity and firmness of an efficient mental or practical activity.

The need for certitude and the history of mathematics

As paradoxical as it may appear, it is mainly as an effect of the scientific endeavor towards rigour, in the history of science, that the rich implications of intuitive knowledge have been revealed and described.

It is by striving to render explicit and to purify the formal, the deductive structure of science that scientists and philosophers have discovered the fundamental effects (both positive and negative) of intuitive mechanisms in understanding, solving, inventing and learning.

The contribution of mathematicians has been the most significant, probably because mathematics, by its very nature, is the most suitable for reaching an axiomatized structure. It is in the course of mathematical thinking that the qualities of a formal, ideal model on one hand and the concrete, psychological constraints on the other, appear so sharply contrasting.

While trying to define the concepts used and to build deductive structures, mathematicians have to take maximum care *not* to rely upon intuitive, implicitly accepted, evidence. Consequently, they have to identify the pitfalls represented by intuitively accepted concepts and statements. The problem of evidence has intervened in the history of mathematics in two important circumstances. Trying to build a deductive, logical structure mathematicians had, first of all, to accept a group of initial statements. The criterion used was that of (apparent) self-evidence: if one has to accept some initial, unproved statements as starting points, it is clear that one tries to choose them among such statements which *may* be accepted without proof. This is what Euclid tried to do when choosing his postulates and axioms.

The second circumstance refers to the efforts made by mathematicians to avoid, so far as possible, the misleading effects of (apparently) evident statements. If one follows the history of mathematics one gets the picture of this dramatic struggle of the human mind towards absolute, unconditional truth. The concern of mathematicians to create a rational, self-consistent, system is already reflected explicitly in the works of the Greeks - mathematicians and philosophers Plato, Aristotle, and Euclid had a clear understanding of the distinction to be made between directly acceptable principles and axioms and those properties which have to be proved. The history of mathematics is, in fact, the history of the endeavours to achieve this programme.

Maintaining intrinsic credibility: an example from the history of physics

As has been discussed above, the historical development of formal structures, with their own form of certitude, does not remove the need for intuitive certitude.

Let us consider for instance, the concept of a *field* in physics. Two bodies, *A* and *B,* exert on each other a force of attraction called gravity. How is this attraction produced? How does the gravitational force propagate itself in space from one point to another? It was assumed, when such influences at a distance were discovered, that some intervening medium must exist which would represent the material support for this propagation. The idea that a body *A* may produce effects on body *B* - and *vice versa* - without any relating medium existing between them, without "something" travelling from one to the other - that idea is *in itself* unacceptable. In a letter addressed by Newton to the Rev. Richard Bentley he clearly expressed this view:

"That gravity should be innate, inherent, and essential to matter so that one body may act upon another at a distance through a vacuum and without the mediation of anything else . . . is to us so great an absurdity that I believe that no man who has in philosophical matters a competent faculty of thinking can ever fall into it". *(cf.* Parsegian, 1968, p. 377).

As a matter of fact, the early interpretations of the field concept assumed the presence of some medium able to convey the influence from one body to another.

Even *after* being taught the accepted view that no intervening medium is necessary, one continues to think about the gravitational interaction as if "something" exists which mediates this transport. One may not be aware of the existence of such an implicit representation but it continues to act tacitly and to influence the ways of reasoning.

The best argument I may produce is the creation of the concept of *field* itself. One accepts nowadays, theoretically, that the force acting between two separate bodies could act *at a distance* without any intervening medium. But such an interpretation *is not credible in itself.* It lacks the convincing evidence of a practical representation. It is the concept of field which replaces the idea of an intervening medium.

Although it seems to eliminate the notion of a travelling or transmitting substratum, the field concept manages to smuggle discreetly a ,behaviorally credible representation into the domain of theoretical physics.

The notion of field certainly gets a theoretical, formal status in physics which, strictly speaking, does not contain any explicit reference to a substantial intervening medium. But psychologically, it has the quality of being intuitively manageable. One may "see" the field - the distance is no longer totally empty. Maxwell has put into it "lines of force" emanating from charges of one polarity and terminating on charges of an opposite polarity. Thinking in pictures is not intuition itself, but it may increase the direct, intrinsic credibility of a concept and this is the basic, natural tendency in every information processing activity even when the corresponding formal structure may appear to be conceptually irreproachable.

The Crisis of Certitude

The scientific community began to realize that self-evidence is not an absolute guarantee of truth, that what today appears as being an absolute property of a category of phenomena may be rejected tomorrow as an incomplete or even incorrect description of the relevant reality. Truth itself has acquired a more and more relativistic connotation.

The Copernican revolution, the non-Euclidean geometries, the special and the general theories of relativity, the findings related to the Cantorian concept of actual infinity, etc. - all these ideas and representations have contributed to the notion that self-evidence (i.e. intuitive evidence) is not synonymous with certainty. More and more non-intuitive or counter-intuitive concepts have invaded science and mathematics. A continuous function without a derivative has no intuitive meaning. The statement that the set of even numbers is equivalent to the set of natural numbers which is equivalent to the set of rational numbers, has not got an intuitive meaning. What is the intuitive meaning of $a^0 = 1$ or of a division like $(2/3) \div (7/12)$?

It is evident that one cannot spend 5 dollars when one has only 3, but mathematically one may write 3 - 5 and ask for a reasonable solution.

For a very long time the concept of actual infinity has been rejected because it leads to the logically unacceptable statement that a set may be equivalent to one of its proper subsets. Since Cantor one accepts the concept of actual infinity and one rejects (or at least has to clarify) the apparently self-evident statement that the whole must always be bigger than each of its parts.

The scientific community started to consider direct evidence - expressed by various intuitively accepted statements and representations - as being potentially misleading. Instead, it was the logical form of certitude which became the ultimate stronghold of scientific conviction.

From its royal position as the absolutely credible form of knowledge (Descartes, Spinoza) intuition has become, for many philosophers, scientists and mathematicians the primitive, the less than credible, the very probably misleading source of knowledge.

If intuition had been only one kind of knowledge among others, its role in a scientific endeavor would have been diminished and finally it would have disappeared from scientific debates, as a problem settled once and for all. But, as we know, this has not happened and intuition reappears time and again, praised or blamed, in philosophical studies, in works devoted to the foundations of mathematics, and in educational theories (See for instance the interesting book of Seymour Papert, 1980).

The simple explanation of this apparently contradictory situation is that intuition - as we have already mentioned - does not represent only a category of knowledge among others, accepted or banned. It expresses a necessary attitude deeply rooted in our adaptive behavior. Mathematicians and scientists continue to discover that concepts which have previously been taken for granted as self-evident have sometimes to be abandoned. Nevertheless they continue to use, consciously or not, intuitive, potentially misleading models. And the debates about the foundations of mathematics continue to refer to the role of the intuitive approach *versus* the formal, strictly deductive one.

The work of Brouwer is extremely significant in this regard. After the emergence of the non-Euclidean geometries, after the essential changes produced by the Cantorian approach, after the fundamental works devoted to the axiomatic method, and despite all these, Brouwer claimed that a true mathematical approach must be intuitionistic!

We shall return to this point later on, but let me cite Hermann Weyl in this respect: “Brouwer”, says Weil, “opened our eyes and made us see how far classical mathematics nourished by a belief in the absolute that transcends all human possibilities of realisation goes beyond such statements that can claim real meaning and truth founded on evidence” (Kline, 1980, p. 235).

And Weyl is no exception. Morris Kline cites a long list of famous mathematicians who, in the last century, emphasized the role of intuitive acceptance in mathematical reasoning (Kline, 1980, pp. 306—327).

Nobody today would claim seriously that it is time to return to Aristotelian physics, to the concept of phlogiston or to the practical geometry of the old Egyptians.

But nobody seems to be really surprised that many mathematicians continue to claim overtly that the final assessment of the validity of a mathematical statement - or even of a proof - is based on the feeling of subjective evidence - exactly as Descartes did three hundred years ago. You may be perfectly aware of what has happened in the history of science and of mathematics to the notion of self-evidence, that many errors have been made on the grounds of absolutized self-evidence, and yet, looking for an ultimate rampart for the defense of your certitudes, in the inextricable mixture of logical arguments you go back to your personal, very personal, feeling of evidence. This kind of intellectual, I would say very honest, duplicity, seems to me to be at the same time absolutely surprising and absolutely natural.